Indirect Proofs

Announcements

- Problem Set 1 goes out today!
- ***Checkpoint*** due Monday, January 14 at 2:30PM.
    - Grade determined by attempt rather than accuracy. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
    - We will get feedback back to you with comments on your proof technique and style.
    - The more effort you put in, the more you'll get out.
- ***Remaining problems*** due Friday, January 18 at 2:30PM.
    - Feel free to email staff list with questions, stop by office hours, or ask questions on Piazza!

Office hours have started!

Schedule is available
on the course website.

Outline for Today

- ***Negations and their Applications***
  - How do you show something is *not* true?
- ***What is an Implication?***
  - *Understanding a key type of mathematical statement.*
  - *Negations of implications.*
- ***Proof by Contrapositive***
  - What's a contrapositive?
  - And some applications!
- ***Proof by Contradiction***
  - The basic method.
  - And some applications!

Negations

Negations

- A ***proposition*** is a statement that is either true or false.
  - Sentences that are questions or commands are not propositions.
- Some examples:
  - If n is an even integer, then n^2 is an even integer.
  - $\emptyset = \mathbb{R}$.
  - Moonlight is a good movie.
- The ***negation*** of a proposition X is a proposition that is true whenever X is false and is false whenever X is true.
- For example, consider the statement "it is snowing outside."
  - Its negation is "it is not snowing outside."
  - Its negation is *not* "it is sunny outside." ⚠

How do you find the negation of a statement?

The negation of the ***universal*** statement

Every *P* is a *Q*

is the *existential* statement

There is a *P* that is not a *Q*.

The negation of the ***universal*** statement

For all *x*, *P*(*x*) is true.

is the *existential* statement

There exists an *x* where *P*(*x*) is false.

The negation of the ***existential*** statement

There exists a *P* that is a *Q*

is the *universal* statement

Every *P* is not a *Q*.

The negation of the ***existential*** statement

There exists an *x* where *P*(*x*) is true

is the *universal* statement

For all *x*, *P*(*x*) is false.

Puppy Logic

- Consider the statement

I love all puppies.

Puppy Logic

- Consider the statement

I love all puppies.

What is the negation?

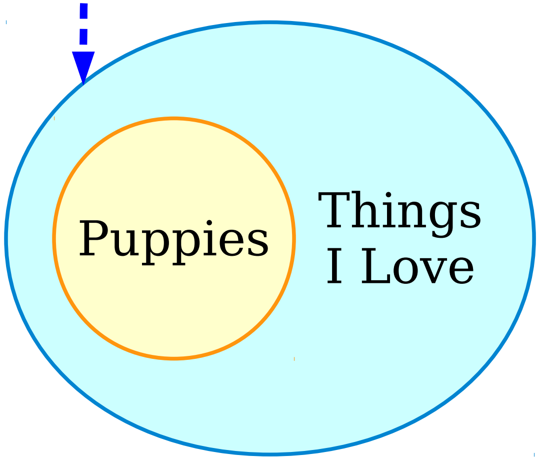
A. I don't love any puppies.
B. I love some puppies and not others.
C. There is at least one puppy I don't love.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, or **C**.

Puppy Logic

- Consider the statement

I love all puppies.



“I love all puppies.”

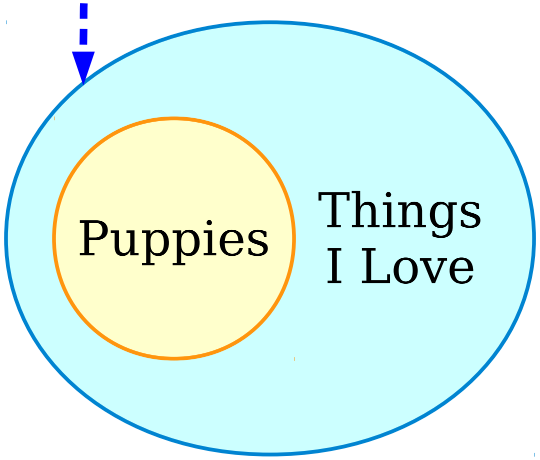
Puppy Logic

- Consider the statement

I love all puppies.

- The following statement is ***not*** the negation of the original statement:

⚠ **I don't love *any* puppies.** ⚠



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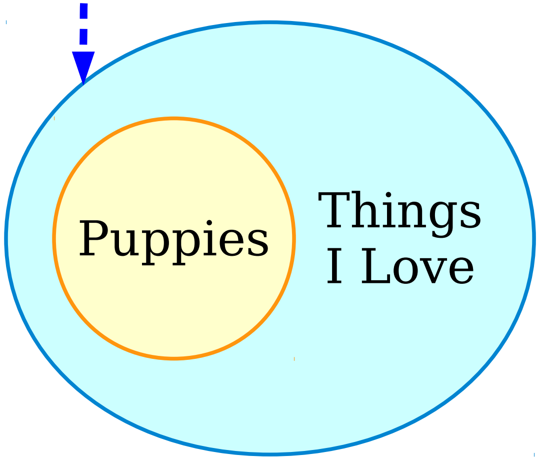

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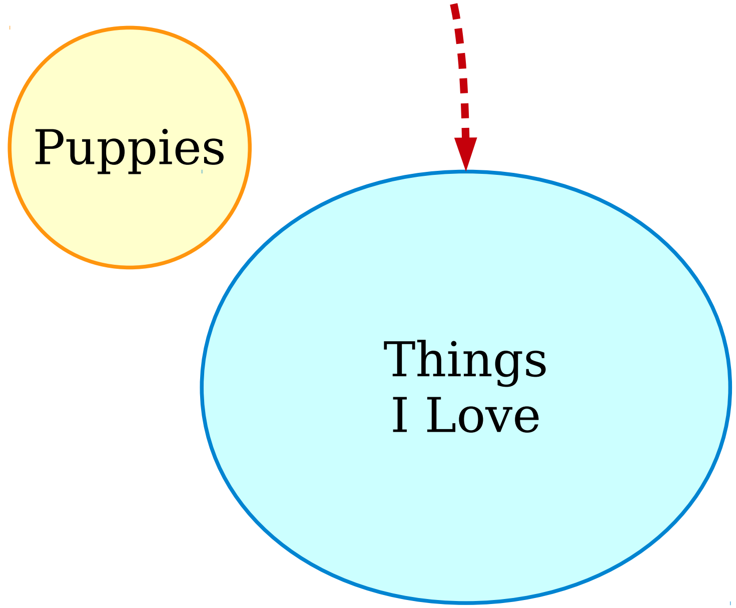

“I don't love *any* puppies.”

Puppy Logic

- Consider the statement

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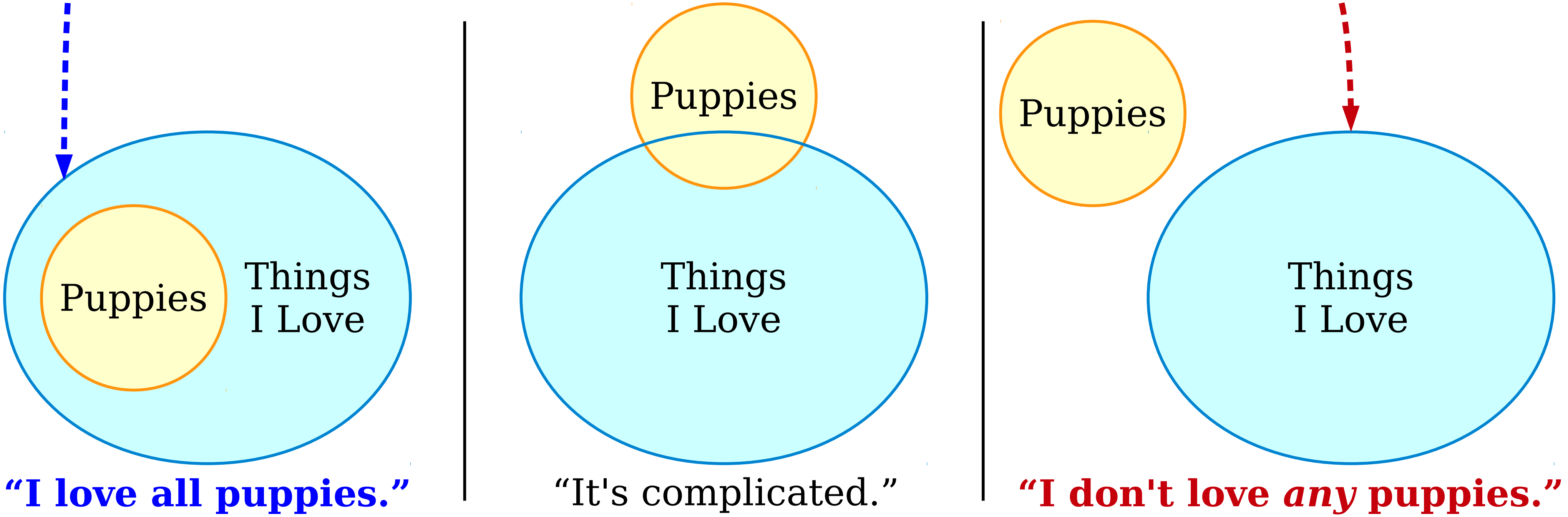


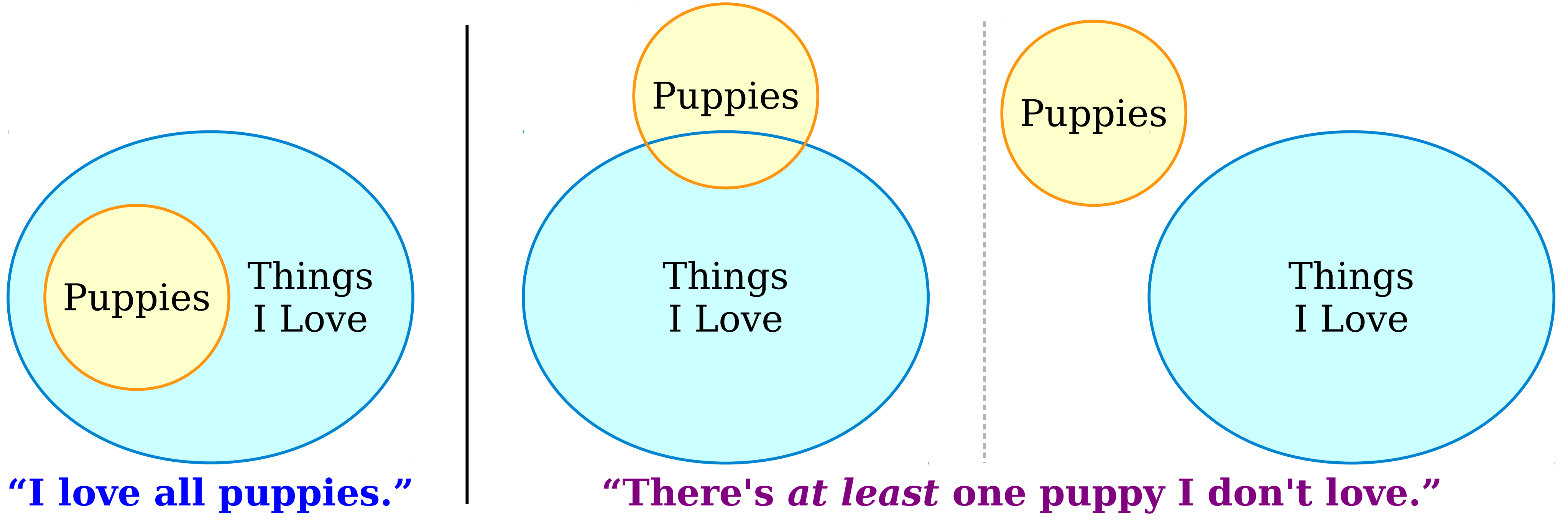
Puppy Logic

- Consider the statement

I love all puppies.

- Here's the proper negation of our initial statement about puppies:

There's *at least* one puppy I don't love.



Logical Implication

Implications

- An ***implication*** is a statement of the form

  If P is true, then Q is true.

- Some examples:
  - Math: If n is an even integer, then n^2 is an even integer.
  - Set Theory: If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
  - Queen Bey: If you like it, then you should put a ring on it.

Implications

“If your March Madness bracket is perfect, then you get an A in CS103.”

Implications as defined in logic/math (this class):

- Implication is *directional*.
  - “If X then Y” is NOT the same as “If Y then X.”
- Implication is *conditional*.
  - It only says something about the consequent when the antecedent is true.
  - If the antecedent is false, “all bets are off.”
- Implication says <u>nothing</u> about *causality*.
  - Simply an assertion about the pattern of T/F occurrences of the antecedent and consequent.

What Implications Mean

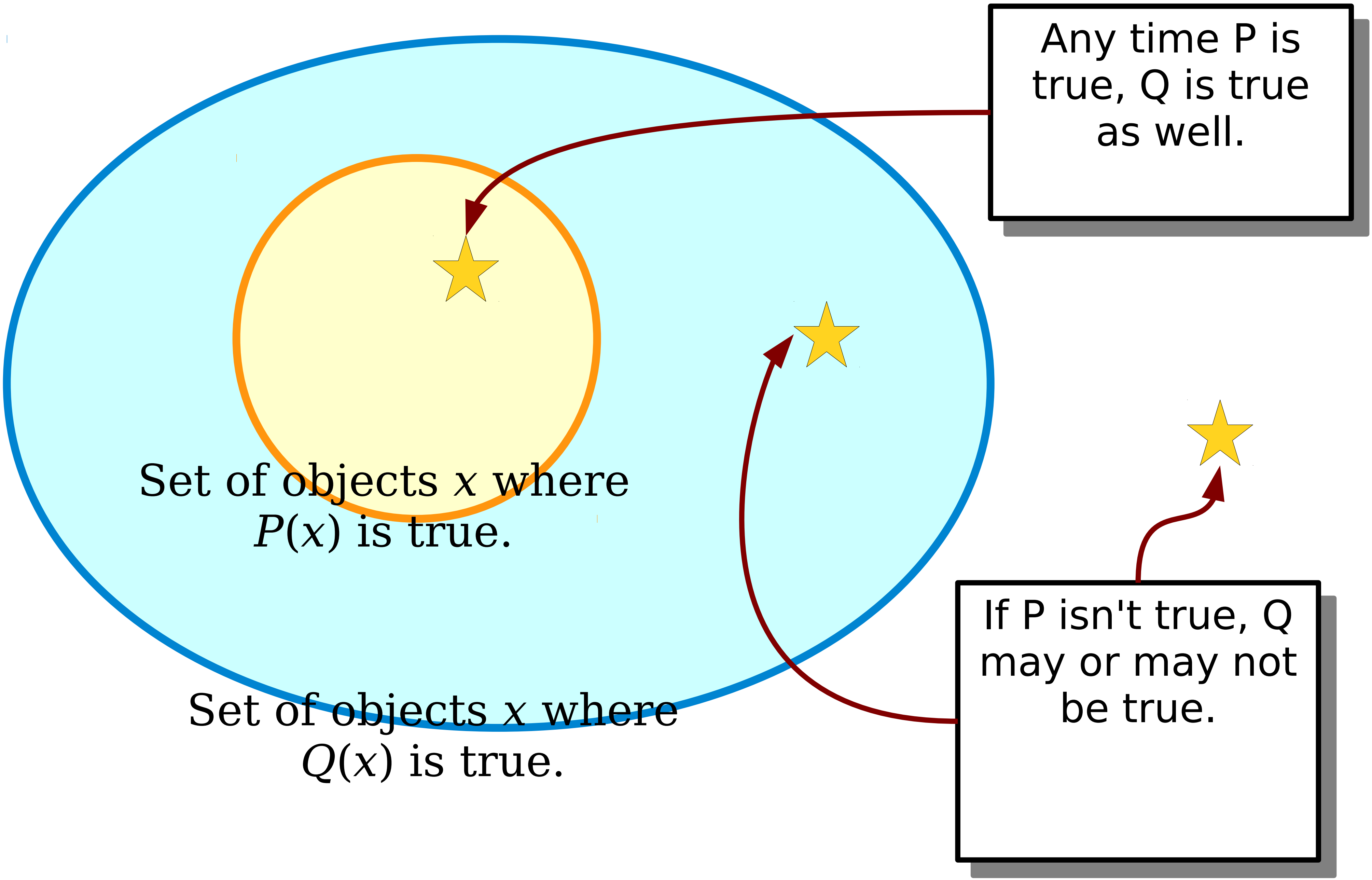
- In mathematics, a statement of the form

  For any x, if $P(x)$ is true, then $Q(x)$ is true

  means that any time you find an object x where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same x).

  - *Reminder:* There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.

Implication, Diagrammatically



Negating Implications

Intuition starter:

"If your March Madness bracket is perfect, then you get an A in CS103."

Which of the following is <u>inconsistent</u> with the above statement?

(A) Your bracket was terrible, and you got an A.
(B) Your bracket was terrible, and you got a B+.
(C) Your bracket was perfect, and you got a B+.
(D) Both (A) and (C)

The negation of the statement

“If *P* is true,
then *Q* is true”

is the statement

“P is true,
and Q is false.”

The negation of an implication
is not an implication!

The negation of the statement

“If *P* is true,
then *Q* is true”

can also be written as

“P is true,
but Q is false.”

The negation of an implication
is not an implication!

We frequently use “but” as a synonym for “and.” “But” carries an emotional connotation of a surprise/unexpected outcome, however from a logic standpoint they are synonyms.

"If your March Madness bracket is perfect, then you get an A in CS103."

Which of the following is <u>inconsistent</u> with the above statement?

(A) Your bracket was terrible, and you got an A.
(B) Your bracket was terrible, and you got a B+.
(C) Your bracket was perfect, and you got a B+.
(D) Both (A) and (C)

“If your March Madness bracket is perfect, then you get an A in CS103.”

Which of the following is <u>inconsistent</u> with the above statement?

(A) Your bracket was terrible, but you got an A.
(B) Your bracket was terrible, but you got a B+.
(C) Your bracket was perfect, but you got a B+.
(D) Both (A) and (C)

Note this is the exact same question, because “but” and “and” are synonyms.

The negation of the statement

“If your March Madness bracket is perfect, then you get an A in CS103.”

is the statement

“Your March Madness bracket is perfect, and you still didn’t get an A in CS103.

The negation of an implication is not an implication!

“Your March Madness bracket is perfect, but you still didn’t get an A in CS103.

Compound Negations

Combining rules for negating an implication and a universal statement

The negation of the statement

"For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

is the statement

"There is at least one *x* where *P*(*x*) is true and *Q*(*x*) is false."

The negation of an implication is not an implication!

Relatives of Negation

Relatives of Negation

- The ***converse*** of the implication

  “If P, then Q”

  is

  “If Q, then P.”

- For example:
  - “If your March Madness bracket is perfect, then you get an A in CS103.”
  - Converse: “If you got an A in CS103 then your March Madness bracket was perfect.”
- **Warning: the converse of a true statement is not necessarily true!**

Relatives of Negation

- The ***inverse*** of the implication

“If P, then Q”

is

“If not P, then *not* Q.”

- For example:
  - “If your March Madness bracket is perfect, then you get an A in CS103.”
  - Inverse: “If your March Madness bracket is not perfect, then you won’t get an A in CS103.”
- **Warning: the inverse of a true statement is not necessarily true!**

The Contrapositive

- The ***contrapositive*** of the implication

  “If P, then Q”

  is

  “If not Q, then not P.”

- For example:
  - “If your March Madness bracket is perfect, then you get an A in CS103.”
  - Contrapositive: “If you didn’t get an A in CS103 then your March Madness bracket wasn’t perfect.”
  - “If you like it, then you should put a ring on it.”
  - Contrapositive: “If you shouldn’t put a ring on it, then you don’t like it.”
- **The contrapositive of a true statement is <u>guaranteed</u> to be true.** *(Thanks, rules of logic!)*

Proof by Contrapositive

To prove the statement

“If *P* is true, then *Q* is true,”

you could choose to instead prove the equivalent statement

“If *Q* is false, then *P* is false.”

(if that seems easier).

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Proof: By contrapositive;

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Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive;

We're starting this proof by telling the reader that it's a proof by contrapositive. This helps cue the reader into what's about to come next.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a safety reinforcement for ourselves by forcing us to write out what we think the contrapositive is (to prevent a careless error in details of contrapositive).

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

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Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Let n be an arbitrary odd integer.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Let n be an arbitrary odd integer. Since n is odd, there is some integer k such that $n = 2k + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: By contrapositive; we prove that if n is odd, then n^2 is odd.

Let n be an arbitrary odd integer. Since n is odd, there is some integer k such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$n^2 = (2k + 1)^2$$

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$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Let n be an arbitrary odd integer. Since n is odd, there is some integer k such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

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From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Let n be an arbitrary odd integer. Since n is odd, there is some integer k such that $n = 2k + 1$. Squaring both sides of this equality and simplifying gives the following:

$$n^2 = (2k + 1)^2$$
$$n^2 = 4k^2 + 4k + 1$$
$$n^2 = 2(2k^2 + 2k) + 1.$$

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Let n be a
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The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

From this
(namely, 2
Therefore

Biconditionals

- Combined with what we saw on Wednesday, we have proven that, if n is an integer:

If n is even, then n^2 is even.

If n^2 is even, then n is even.

- Therefore, if n is an integer:

n is even if and only if n^2 is even.

- "If and only if" is often abbreviated ***iff***:

n is even iff n^2 is even.

Proving Biconditionals

- To prove a theorem of the form ***P* iff *Q***, you need to prove that *P* implies *Q* and that *Q* implies *P*. (two separate proofs)
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof for one and a proof by contrapositive for the other.

Proof by Contradiction

“When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.”

- Sir Arthur Conan Doyle*, The Adventure of the Blanched Soldier*

Proof by Contradiction

- A ***proof by contradiction*** is a proof that works as follows:
  - To prove that P is true, assume that P is *not* true.
  - Beginning with this assumption, use logical reasoning to conclude something that is clearly impossible.
    - For example, that $1 = 0$, that $x \in S$ and $x \notin S$, etc.
  - This means that if P is false, something that cannot possibly happen, happens!
  - Therefore, P can't be false, so it must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We’ve seen sets of many different cardinalities:
  - $|\emptyset| = 0$
  - $|\{1, 2, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137\}| = 137$
  - $|\mathbb{N}| = \aleph_0$.
- These span from the finite up through the infinite.
- ***Question:*** Is there a “largest” set? That is, is there a set that’s bigger than every other set?

Theorem: There is no largest set.

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Proof:

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Proof:

To prove this statement by contradiction, we're going to assume its negation.

Theorem: There is no largest set.

Proof:

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

Theorem: There is no largest set.

Proof:

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

One option: **"there *is* a largest set."**

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

To prove this statement by contradiction, we're going to assume its negation.

What is the negation of the statement "there is no largest set?"

One option: **"there *is* a largest set."**

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

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Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Notice that we're announcing

1. that this is a proof by contradiction, and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

Now, consider the set $\wp(S)$. By Cantor's Theorem, we know that $|S| < |\wp(S)|$, so $\wp(S)$ is a larger set than S. This contradicts the fact that S is the largest set.

We've reached a contradiction, so our assumption must have been wrong.

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ■

Proving Implications

- To prove the implication

"If *P* is true, then *Q* is true."

- you can use these three techniques:
  - ***Direct Proof.***
    - Assume P and prove Q.
  - ***Proof by Contrapositive.***
    - Assume not Q and prove not P.
  - ***Proof by Contradiction.***
    - … what does this look like?

Theorem: If n is an integer and n^2 is even, then n is even.

Theorem: If n is an integer and n^2 is even, then n is even.
Proof: Assume for the sake of contradiction that _______

What is the assumption?

A. if n is odd, then n^2 is odd
B. n is an integer and n^2 is even, and n is odd
C. if n is an integer and n^2 is odd, then n is odd
D. n is an integer and n^2 is odd, and n is odd

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, or **C**.

Theorem: If n is an integer and n^2 is even, then n is even.
Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

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Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

Since n is odd we know that there is an integer k such that

$$n = 2k + 1. \tag{1}$$

Theorem: If n is an integer and n^2 is even, then n is even.

Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

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$$n = 2k + 1. \tag{1}$$

Squaring both sides of equation (1) and simplifying gives the following:

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Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

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Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned} \tag{2}$$

Theorem: If n is an integer and n^2 is even, then n is even.
Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

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Squaring both sides of equation (1) and simplifying gives the following:

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Equation (2) tells us that n^2 is odd, which is impossible; by assumption, n^2 is even.

Theorem: If n is an integer and n^2 is even, then n is even.
Proof: Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.

Since n is odd we know that there is an integer k such that

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what the negation of the original statement is.
3. Say you have reached a contradiction and what the contradiction entails.

In CS103, please include all these steps in your proofs!

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Theorem: There is no greatest integer.
Proof: Assume for the sake of contradiction that there is a greatest integer, and call it n.

What is the "want to show"?

A. there is no greatest integer
B. n is the greatest integer, which is a contradiction
C. find any two facts that are the negation of each other, which is a contradiction
D. Other

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, or **C**.

Recap: Negating Implications

- To prove the statement

  “For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true”

  by contradiction, we do the following:
  - Assume this entire purple statement is false.
  - Derive a contradiction.
  - Conclude that the statement is true.
- What is the negation of the above purple statement?

  “There is an *x* where
  ***P*(*x*) is true and *Q*(*x*) is false”**

Recap: Contradictions and Implications

- To prove the statement

  "If P is true, then Q is true"

  using a proof by contradiction, do the following:

  - Assume that P is true and that Q is false.
  - Derive a contradiction.
  - Conclude that if P is true, Q must be as well.

What We Learned

- ***What's an implication?***
  - It's statement of the form "if P, then Q," and states that if P is true, then Q is true.
- ***How do you negate formulas?***
  - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of "if P, then Q" is "if not Q, then not P.")
- ***What's a proof by contradiction?***
  - It's a proof of a statement P that works by showing that P cannot be false.

Next Time

- ***Mathematical Logic***
  - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
  - Reasoning about simple statements.
- ***Propositional Equivalences***
  - Simplifying complex statements.